

# Strings in gravity with torsion

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## Abstract

A theory of gravitation in 4D is presented with strings used in the material action in  $U_4$  spacetime. It is shown that the string naturally gives rise to torsion. It is also shown that the equation of motion a string follows from the Bianchi identity, gives the identical result as the Noether conservation laws, and follows a geodesic only in the lowest order approximation. In addition, the conservation laws show that strings naturally have spin, which arises not from their motion but from their one dimensional structure.

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## 1 Introduction

Two branches in general relativity have met with varying degrees of success over the years. One is the use of strings to model, or reflect, the structure of matter. This topic, while still relatively young, has provided substantial hope into a successful program of quantum gravity,<sup>1</sup> and generates great interest at the classical level as well.<sup>2</sup> The other branch reaches further back and modifies Einstein's original theory by taking the affine connection to be asymmetric. The antisymmetric part of the affine connection — the torsion, has a twisted history, but gained renewed popularity in the 1970s when it was used in developing a local Poincaré gauge theory of gravity.<sup>3</sup> The interest in torsion gained a wider audience when

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it was shown that it could act as the antisymmetric field that is required in string theory,<sup>4</sup> and gained further notice when it appeared in supergravity theories.<sup>5</sup> The two once disparate branches have become intertwined, and the purpose here is to tap into the synergism generated by this growth.

One of the underlying bonds that seems to unite string theory and general relativity with torsion is the assumption that the torsion may be derived from an antisymmetric potential  $\psi_{\mu\nu}$  according to

$$S_{\mu\nu\sigma} = \psi_{[\mu\nu,\sigma]}. \quad (1)$$

When this assumption was used to develop a theory of gravitation with torsion, the physical interpretation that fell upon torsion was that it was created from intrinsic spin.<sup>6</sup> The interpretation resulted from an analysis of the conservation laws of the equations of motion. Some of the salient features that were discussed and shown in [6] are the following. It was shown that it was necessary to introduce an intrinsic vector  $\xi^\mu$  to represent the source for torsion. This may be viewed as a generalization of other intrinsic quantities used in describing sources. For example, to describe a point particle we may consider its mass  $m$  as an intrinsic quantity, or its charge  $q$  as an intrinsic quantity. The source tensors are built up from these quantities. In order to couple a source to an antisymmetric field, as is the case for torsion, it is necessary to go beyond scalar intrinsic quantities and adopt the intrinsic vector approach. This gave rise to several aspects of the theory. One is that intrinsic spin is derived from this intrinsic vector. Thus, intrinsic spin arises, not from motion or rotation, but from structure and persists in the limit of a static configuration. Another result was that the particle had to have structure, and labeling each point on the structure by a vector  $\xi_n^\mu$ , it was shown that the conservation laws imply  $\sum_n \xi_n^\mu = 0$ . The conservation laws also predicted a definite spin interaction term, that in principle, could be observed. This interaction term was confirmed when the Dirac Lagrangian was used in place of the intrinsic vector material action. In the low energy limit, the Dirac equation yielded that same interaction as that predicted by the intrinsic vector approach,<sup>7</sup> and this was used to place an upper bound on the coupling constant.<sup>8</sup> In addition, it was shown that a scalar arises naturally,<sup>9</sup> and it was also shown that this scalar field might be interpreted as the scalar field of string theory.<sup>10</sup>

Thus, the theory of gravitation with torsion with (1) adopts many characteristics of string theory — from the mirrored Lagrangian of the low energy string theory limit, to a source which must be at least one dimensional. It is natural then to consider replacing the intrinsic vector material Lagrangian by that of a string. In this case, many plausible results follow. First, the intrinsic vector will be seen to correspond to the tangent vector of the string. Also the condition given above that the intrinsic vectors sum to zero, is replaced by the much more natural result  $\oint d\zeta = 0$ , which is true for closed strings. Moreover, there is a natural coupling between the torsion potential and the area swept out by the

string. It will be shown that many of the results of the intrinsic vector approach will be reproduced by use of the string source, and in fact, in many ways, do so in a much more natural fashion. It will also be shown that strings have intrinsic spin, which follows from their structure (not motion), and the equations of motion will follow from the conservation laws. In what follows only closed strings will be considered, the discussion of open strings will be considered separately. Finally, I would like to give one more preview of things to come. It is customary to obtain the equation of motion of a string by adopting the geodesic postulate, i.e., setting the variations of the action with respect to the coordinate equal to zero, or in a more rigorous approach, using invariance of the material action.<sup>11</sup> However, in general relativity the equations of motion of an object follow from the field equations, and variations with respect to the coordinate will not be considered. Thus, certain conditions or properties of the string well known from string theory are not necessarily appropriate to this case.

## 2 Field equations and the conservation laws

### 2.1 Field equations

The material in this section is general in the sense that no explicit form of the energy momentum tensor is given. Details on this material may be found in [6]. The field equations are given by the action principle

$$\delta(I_g + I_m) = 0 \quad (2)$$

where

$$I_g = \int \sqrt{-g} \frac{R}{2k} d^4x, \quad (3)$$

$I_m$  is the material action, and  $k = 8\pi$ . The curvature scalar  $R$  is that of  $U_4$  spacetime, and therefore contains torsion. The unknown quantities are the 10 components of the symmetric metric tensor and the 6 components of the antisymmetric torsion potential. The equations are obtained by considering independent variations of these potentials. (One may note that this action principle yields second order differential equations in the torsion potential, whereas an action given by (3), but with variations taken with respect to the torsion itself, yields non-propagating torsion.) However, an equivalent procedure is, defining  $\phi_{\mu\nu} = g_{\mu\nu} + \psi_{\mu\nu}$ , to consider variations with respect to  $\phi_{\mu\nu}$ . Now, taking variations with respect to  $\phi_{\mu\nu}$  gives a set of 16 equations. The symmetric part is equivalent to those obtained by considering variations with respect to the metric tensor, and will be called the gravitational field equations. The antisymmetric part is equivalent to the equations obtained by

performing variations with respect to the torsion potential and will be called the torsional field equations. In this case, a non-symmetric energy momentum tensor was defined (see [6]) according to

$$\delta I_m = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta \phi_{\mu\nu}. \quad (4)$$

The symmetric part of the energy momentum tensor in (4) is the source in the gravitational field equations, and the antisymmetric part is the source in the torsional field equations. With these remarks, the field equations are given by

$$G^{\mu\nu} - 3S^{\mu\nu\sigma}{}_{;\sigma} - 2S^{\mu}{}_{\alpha\beta} S^{\nu\alpha\beta} = kT^{\mu\nu} \quad (5)$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \quad (6)$$

and  $R^{\mu\nu}$  is the (asymmetric) Ricci tensor in  $U_4$  spacetime. In line with the above remarks, the torsional field equations are given by

$$S^{\mu\nu\sigma}{}_{;\sigma} = -k j^{\mu\nu} \quad (7)$$

where  $j^{\mu\nu} \equiv (1/2)T^{[\mu\nu]}$ . In the above, and below, it is convenient to use two different kinds of covariant differentiation. First, the fundamental definition of a covariant derivative is given by, for any vector  $A^\mu$ ,

$$\nabla_\sigma A^\mu = A^\mu{}_{;\sigma} + \Gamma_{\sigma\nu}{}^\mu A^\nu \quad (8)$$

which contains the full (asymmetric) affine connection  $\Gamma_{\sigma\nu}{}^\mu$ . However, sometimes the antisymmetric part drops out, and it is useful to define the ‘‘Christoffel’’ covariant derivative by

$$A^\mu{}_{;\sigma} = A^\mu{}_{,\sigma} + \{\sigma\nu\}^\mu A^\nu \quad (9)$$

where  $\{\sigma\nu\}^\mu$  is the (symmetric) Christoffel symbol. The relation between the affine connection and the Christoffel symbol is obtained by the requirement  $\nabla_\sigma g_{\mu\nu} = 0$ , which yields

$$\Gamma_{\mu\nu}{}^\sigma = \{\sigma\mu\nu\} + S_{\mu\nu}{}^\sigma + S_{\mu\nu}{}^\sigma + S_{\nu\mu}{}^\sigma \quad (10)$$

which becomes, with (1),

$$\Gamma_{\mu\nu}{}^\sigma = \{\sigma\mu\nu\} + S_{\mu\nu}{}^\sigma. \quad (11)$$

## 2.2 Conservation laws

Ultimately, the correctness of any theory can only be ascertained by experiment. In a theory of gravitation this means that one must derive the equations of motion, which predict acceleration, interaction energy or something that is measurable. Not only do the equations of motion provide the means to evaluate the theory, they are also instrumental in developing the physical interpretation of the theory. A powerful aspect of general relativity is that the equations of motion follow from the field equations, and therefore already establish the machinery for the analysis of the predictions of the theory. In fact, this comes about in two ways, each of which yields the same result. One is that the Bianchi identities must be obeyed, which are, in  $U_4$  spacetime,

$$\nabla_\nu G^{\mu\nu} = 2S^{\mu\alpha\beta}R_{\beta\alpha} - S_{\alpha\beta\gamma}R^{\mu\gamma\beta\alpha}. \quad (12)$$

To use the Bianchi identities, one operates with  $\nabla_\nu$  on (5) and uses (12). This establishes a differential relation on the source. The other way to proceed is to capitalize on the requirement that the material action is a scalar, and that under the manifold mapping  $x^\mu \rightarrow x^\mu + \epsilon^\mu$ ,

$$\int d^4x \sqrt{-g} T^{\mu\nu} \mathcal{L}_\epsilon \phi_{\mu\nu} = 0 \quad (13)$$

where  $\mathcal{L}_\epsilon$  is the Lie derivative. Either approach yields

$$T^{\mu\nu}{}_{;\nu} = \frac{3}{2} T^{\alpha\beta} S^\mu{}_{\alpha\beta}. \quad (14)$$

This result shows that when the torsion vanishes, we obtain the conventional result that the covariant derivative of the energy momentum vanishes. The equations of motion then follow from this result. With torsion, (14) shows that there will be additional forces due to torsion, and that geodesic motion should not be expected.

## 3 Equations of motion

### 3.1 The general case

The equations of motion can be found using the method of Papapetrou.<sup>12</sup> At first, the method will be kept general, meaning that the actual source will not be specified. After that, the formulation will be used to find the equation of motion of a small string in an external field. Some general comments about this method are: It is not generally covariant. Volume integrals over the test object are considered at constant  $x_0$ . Also, under consideration is the

motion of a small test object in the presence of a large object. The gravitational field of the test object is essentially ignored, except in the way the inertial mass is defined. These notions will be clarified as we go.

The starting point is the identity

$$\tilde{T}^{\mu\nu}_{;\nu} = \sqrt{-g}T^{\mu\nu}_{;\nu} - \{\alpha\beta\}^{\mu}\tilde{T}^{\alpha\beta} \quad (15)$$

where the tilde implies density according to  $\tilde{T}^{\mu\nu} = \sqrt{-g}T^{\mu\nu}$ . Now, consider a small volume  $d^3x$  that completely enclosed the test object, and integrate this equation over that volume. In that region, the energy momentum tensor of the large body is zero, so from here on the energy momentum tensor is that of the small body—the string. In other words, we suppose that

$$I_m = I_b + I_s \quad (16)$$

where  $I_b$  is the material action corresponding to the energy momentum tensor of the big body, and  $I_s$  that of the small body so that

$$\delta I_s = \frac{1}{2} \int d^4x \sqrt{-g} T_s^{\mu\nu} \delta \phi_{\mu\nu}, \quad (17)$$

and from here on the subscript  $s$  is dropped.

The metric tensor that appears in these formulas is total gravitational field of both objects, however, assuming that the large object has much more mass than the small object, the metric tensor that appears in (15) is approximated by that of the large body. Discarding surface terms, and using the conservation law (14), (15) becomes

$$\frac{d}{dx^0} \int \tilde{T}^{\mu 0} = \frac{3}{2} \int \tilde{T}^{\alpha\beta} S_{\alpha\beta}^{\mu} - \int \tilde{T}^{\alpha\beta} \{\alpha\beta\}^{\mu} \quad (18)$$

where the volume element has been, and will be, suppressed. The following definitions will be useful, with  $\tau^{\alpha\beta} \equiv \tilde{T}^{(\alpha\beta)}$ :

$$M^{\mu\nu} = v^0 \int \tilde{T}^{\mu\nu}, \quad (19)$$

$$M^{\alpha\mu\nu} = -v^0 \int \delta x^{\alpha} \tau^{\mu\nu} \quad (20)$$

$$m^{\alpha\mu\nu} = v^0 \int \delta x^{\alpha} \tilde{j}^{\mu\nu} \quad (21)$$

$$J^{\mu\nu} = \int (\delta x^{\mu} \tilde{T}^{\nu 0} - \delta x^{\nu} \tilde{T}^{\mu 0}). \quad (22)$$

We consider  $x^\alpha$  as the coordinate from the origin to a point on the small body. Then  $y^\alpha$  is defined according to  $x^\alpha = y^\alpha + \delta x^\alpha$ , where  $\delta x^\alpha \ll y^\alpha$ . To proceed, the Cristoffel symbols and the torsion tensor are expanded in a Taylor series about the point  $y^\mu$ . These quantities may be then taken outside the integrals, and using the above equations one may show that

$$\frac{d}{d\tau} \left( \frac{M^{\mu 0}}{v^0} \right) + \{\overset{\mu}{\alpha\beta}\} M^{\alpha\beta} = \{\overset{\mu}{\alpha\beta}\}_{,\eta} M^{\eta\alpha\beta} + \frac{3}{2} M^{\alpha\beta} S^\mu_{\alpha\beta} + 3 S^\mu_{\alpha\beta,\eta} m^{\eta\alpha\beta}. \quad (23)$$

In the appendix it is shown that we may take

$$\int \tilde{j}^{\mu\nu} = 0. \quad (24)$$

With this

$$p^\mu \equiv \frac{M^{\mu 0}}{v^0} = \int \tau^{\mu 0} dV, \quad (25)$$

and (23) becomes

$$\frac{dp^\mu}{d\tau} + \{\overset{\mu}{\alpha\beta}\} M^{\alpha\beta} = \{\overset{\mu}{\alpha\beta}\}_{,\eta} M^{\eta\alpha\beta} + 3 S^\mu_{\alpha\beta,\eta} m^{\eta\alpha\beta}. \quad (26)$$

If the torsion vanishes, this equation reduces to the equation derived by Papapetrou many years ago. The first term on the right side represents the force on a particle with structure due the gradient of the field. This gives rise to the well know result (still no torsion) that only point particles with no structure follow along geodesics. Any structure to a particle will give rise to  $M^{\alpha\mu\nu}$ , which by (26) gives rise to non-geodesic motion. Thus we anticipate that strings, even in Riemannian space, will not follow geodesic motion. The second term on the right side of (26) is the force on the particle due to the torsion. It was shown that when the torsion was constructed from the intrinsic vector, this force was due to the interaction of the intrinsic spin of the particle and the external torsion field.

The Papapetrou method also gives rise to the equation for angular momentum by starting with the identity

$$(x^\alpha \tilde{T}^{\beta\gamma})_{,\gamma} = \tilde{T}^{\beta\alpha} + x^\alpha \tilde{T}^{\beta\gamma}_{,\gamma}, \quad (27)$$

which gives

$$\int \tilde{T}^{\beta\alpha} = \frac{d}{dx^0} \int (y^\alpha + \delta x^\alpha) \tilde{T}^{\beta 0} - \int (y^\alpha + \delta x^\alpha) \tilde{T}^{\beta\gamma}_{,\gamma}. \quad (28)$$

Using the same kinds of manipulations as above this gives

$$\frac{d}{d\tau} J^{\alpha\beta} + \frac{dy^\alpha}{d\tau} \frac{M^{\beta 0}}{v^0} - \frac{dy^\beta}{d\tau} \frac{M^{\alpha 0}}{v^0} = 6 S^{[\beta}_{\mu\nu} m^{\alpha]\mu\nu} + 2 \{\overset{[\beta}{\mu\nu}\} M^{\alpha]\mu\nu}. \quad (29)$$

In the limit that the gravitational and torsional field go to zero the right side of this equation becomes zero. In addition, in this limit, (26), with (25), show that  $M^{\alpha 0}/v^0$  is the constant momentum. In this case (29) yields, upon integration,

$$J^{\alpha\beta} + y^\alpha p^\beta - y^\beta p^\alpha = \text{constant}. \quad (30)$$

This shows that, since the second two terms on the left represent the orbital angular momentum,  $J^{\alpha\beta}$  must represent the total rotational angular momentum plus intrinsic spin. With the torsion set equal to zero, these results are identical to Papapetrou's results. The main differences we are about to encounter below are that torsion is not zero, and the energy momentum tensor is not symmetric, as explained above. In fact, one may see already that the non-symmetric part of the energy momentum tensor enters into  $J^{\alpha\beta}$  through its definition, and therefore we see in general, from this result, that the antisymmetric part of the energy momentum tensor represents intrinsic spin.

## 3.2 Enter the string

Now we would like to consider that the energy momentum tensor that describes the particle discussed above is that of a string, so that, for a simple Nambu-Goto string we assume

$$I_s = \mu \int \sqrt{-\gamma} d^2\zeta + \frac{\mu\eta}{2} \int \sqrt{-\gamma} \psi_{\mu\nu} d\sigma^{\mu\nu} \quad (31)$$

where

$$d\sigma^{\mu\nu} = \epsilon^{ab} x_{,a}^\mu x_{,b}^\nu d^2\zeta \quad (32)$$

and  $\sqrt{-\gamma}\epsilon^{10} = 1$ , etc. The first term is the usual Nambu-Goto action, and is the conventional way in which the string is introduced into gravity. In order to couple something to the torsion potential  $\psi_{\mu\nu}$ , an antisymmetric source term must be invoked. A very natural choice arises with strings, and this is to couple the torsion potential to the worldsheet area element, as is done above. This coupling was first used by Kalb and Ramond.<sup>13</sup> The string coordinates are labeled by  $\zeta^a$  where  $a$  and  $b$  range from 0 to 1. Due to the coordinate invariance of the string action we may choose  $\zeta^0 = x^0$ , and call  $\zeta^1 = \zeta$ . In curved space, we are not allowed therefore to choose the conformal gauge, and we won't. Using this and the definition (17) one obtains

$$T^{\sigma\nu} = \frac{\mu}{\sqrt{-g}} \int d^2\zeta \sqrt{-\gamma} \delta(x - x(\zeta)) x_{,a}^\sigma x_{,b}^\nu (\gamma^{ab} + \eta\epsilon^{ab}) \quad (33)$$

where

$$\gamma_{ab} = x_{,a}^\mu x_{,b}^\nu g_{\mu\nu}. \quad (34)$$



It is worth emphasizing some differences that arise between these equations and those that appear in string theory or in the study of cosmic strings. First, of course, this is a completely classical presentation. However, we do not impose that variations of the string action with respect to the coordinate vanish. Thus, the common ‘geodesic’ condition that is often imposed on the string coordinates, which in curved space this disallows a static rigid string (from the Nambu-Goto action), is not enforced. The equation of motion of the string is derived below from the Bianchi identity. Also, since we chose  $x^0 = \zeta^0$ , we are not able to arbitrarily choose a gauge for the string metric  $\gamma^{ab}$ . In this case, we are considering the equation of motion of the string in an external field, so  $\gamma^{ab}$  is determined from this external field according to (34). In fact, from (34) we see that  $\gamma_{00} = (v^0)^{-2}$ . We also see that  $\gamma_{01} = g_{01}\partial x^1/\partial\zeta$ . Assuming that  $g_{01} \ll g_{00}, g_{11}$ , we may ignore the off diagonal components of the two dimensional metric. Finally, we see that  $\gamma_{11} = g_{mn}(dx^m/d\zeta)(dx^n/d\zeta)$  where  $m, n = 1-3$ . To calculate this we assume that  $g_{mn}$  in the last equation can be replaced by  $\eta_{mn}$ . To justify this, we should examine the equation of motion (26) where this will be used. If we limit the discussion to weak fields so that  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  where  $h_{\mu\nu} \ll \eta_{\mu\nu}$ , then we may be content to carry only terms linear in  $h_{\mu\nu}$  in the equation of motion. Now, since  $T^{\mu\nu}$  (through the quantities  $M^{\mu\nu}$  etc.), which contains  $\gamma_{ab}$ , is multiplied by  $g_{\mu\nu}$  and its derivatives (and products), to this order it is sufficient to replace  $g_{\mu\nu}$  by  $\eta_{\mu\nu}$  in (33). With this, in the appendix it is shown that  $\gamma_{11} = -1$ , so that the energy momentum tensor of the string becomes

$$\tilde{T}^{\alpha\beta} = \frac{\mu}{v^0} \int d^2\zeta \delta(x - x(\zeta)) [(v^\alpha v^\beta - x'^\alpha x'^\beta) + \eta(v^\beta x'^\alpha - v^\alpha x'^\beta)] \quad (35)$$

where  $v^\alpha = dy^\alpha/d\tau$  is the four velocity of the center of mass and  $x'^\alpha = dx^\alpha/d\zeta$ .

With this, we can put the equation of motion (26) can be put into a more recognizable, or useful, form. For the present we will content ourselves to obtain the equation of motion in the lowest order. This means that the terms of the right hand side of (26) will be neglected for now. A more detailed examination of these terms will be reserved for future work. With (35) one obtains

$$M^{\alpha\beta} = \mu \int d\zeta (v^\alpha v^\beta - x'^\alpha x'^\beta). \quad (36)$$

With this, (26) becomes

$$\frac{d}{d\tau}(mv^\sigma) + \{\alpha_\beta^\sigma\} M^{\alpha\beta} = 0. \quad (37)$$

where

$$m \equiv \frac{M^{00}}{(v^0)^2} = \mu \int d\zeta. \quad (38)$$

Now, (37) becomes

$$\frac{dm}{d\tau} v^\sigma + m \frac{dv^\sigma}{d\tau} + \{\sigma_{\alpha\beta}\} \left( m v^\alpha v^\beta - \mu \int d\zeta x'^\alpha x'^\beta \right) = 0. \quad (39)$$

Also defining

$$\frac{Dv^\sigma}{d\tau} = \frac{dv^\sigma}{d\tau} + \{\sigma_{\mu\nu}\} v^\mu v^\nu, \quad (40)$$

we can use the identity  $v_\sigma Dv^\sigma/d\tau = 0$ , so that (39) gives

$$\frac{dm}{d\tau} = \mu v_\sigma \{\sigma_{\alpha\beta}\} \int d\zeta x'^\alpha x'^\beta \quad (41)$$

which puts us in the momentarily awkward position that, even neglecting the terms on the right side of (26), we do not have geodesic motion and the inertial mass is not conserved. However, consider

$$\begin{aligned} \int_0^L d\zeta x'^\alpha x'^\beta &= \int dx^\alpha x'^\beta = - \int x^\alpha d(x'^\beta) = \\ &- \int (y^\alpha + \delta x^\alpha) d(x'^\beta) \approx -y^\alpha \int d(x'^\beta) = 0 \end{aligned} \quad (42)$$

where integration by parts was used and it is assumed that *the string is a closed loop* so that the end point terms contribute nothing, and the last step follows for a closed loop that has no kinks. This shows, finally, the inertial mass is conserved and that (39) reduces to

$$\frac{dv^\sigma}{d\tau} + \{\sigma_{\alpha\beta}\} v^\alpha v^\beta = 0. \quad (43)$$

This result is received as good news for several reasons. First, of course, the inertial mass is conserved. Second, in this lowest order approximation, the string moves along a geodesic. We also see that the structure of the string will cause its actual motion to deviate from the geodesic, and these effects can be calculated from (26). These results also show that a sensible equation of motion results from the Bianchi identity (or from the conservation laws) as it should. Thus, there is no need to make the additional “geodesic” postulate, that variations of the matter action with respect to the coordinates vanish. In fact, they probably do not.

To understand better the physical significance of this energy momentum tensor, and in particular to see that this implies that strings have intrinsic spin, we start from the spin vector, defined by

$$S_\gamma = \frac{v^\sigma}{2} J^{\alpha\beta} \epsilon_{\sigma\alpha\beta\gamma}, \quad (44)$$

which for low velocities becomes

$$S_\gamma \rightarrow \frac{1}{2} J^{\alpha\beta} \epsilon_{0\alpha\beta\gamma}, \quad (45)$$

and make the same approximations for  $T^{\alpha\beta}$  as we did above. When (35) is used in (22), which is used in (44), two kinds of terms arise, those that depend on velocity (3-velocity) and those that do not. The velocity dependent terms represent a conventional  $\mathbf{r} \times \mathbf{p}$  angular momentum of string, about its center of mass, due to its oscillations or rotations. If we restrict our attention to the rigid loop, these terms vanish. Even if the string is not rigid, but is small, i.e., elementary particle size, this term is negligible. Thus, we restrict our attention to the terms that do not depend on velocity, and obtain, for the circular loop,

$$S_\gamma = 2 \int d\zeta \epsilon_{0\alpha\beta\gamma} \delta x^\alpha \frac{dx^\beta}{d\zeta}, \quad (46)$$

or

$$\mathbf{S} = \mu\eta \int \mathbf{r} \times \mathbf{r}' d\zeta \Rightarrow S = 2\mu\eta \times \text{Area} \quad (47)$$

where  $\mathbf{r}' = d\mathbf{r}/d\zeta$ , and  $\mathbf{r}$  is a vector from the center the center of mass to a point on the string.

This result shows that strings, due to the fact that they have structure, give rise to intrinsic spin when they are coupled to the torsion potential. (One may note that the conventional method of discussing a spinning string in string theory is described by the introduction the Grassmannian  $\psi$ , leading to the Raymond model or the Neveu-Schwarz.<sup>14</sup> Here, we will have intrinsic spin from the Nambu-Goto action alone, due only to its structure.) Now we may turn our attention to the torsion field they generate.

## 4 Torsion from a string

The adoption of the string action (31) has two significant consequences. One, as shown above, ties intrinsic spin to the string. The other consequence, and in fact the original motivation in adopting (31), is that it acts a source of torsion. In this section, we shall consider the Minkowski limit, and solve the torsional field equations in this limit. Of course, since torsion enters in the gravitational field equations, strictly speaking, torsion cannot exist without a concomitant gravitational field. However, as has been shown, when torsion arises

from intrinsic spin the torsion field is very small and Minkowski spacetime is a very good approximation.

It is helpful to use the dual to torsion

$$b_\mu = \epsilon_{\mu\alpha\beta\gamma} S^{\alpha\beta\gamma} \quad (48)$$

and  $\mathbf{b} = (b_n)$ . With this in mind (7) becomes

$$\square \psi^{\mu\nu} = -3k j^{\mu\nu}. \quad (49)$$

Now we consider the static case, so that this reduces to

$$\nabla^2 \mathbf{A} = \mathbf{N} \quad (50)$$

where

$$A_n \equiv 2\psi_{0n}, \quad \mathbf{A} = (A_n) \quad (51)$$

and

$$N^n \equiv 6k j^{0n}, \quad \mathbf{N} = (N^n) \quad (52)$$

and a ‘Lorentz’ gauge is chosen so that

$$\psi^{\sigma\mu}_{,\sigma} = 0, \quad (53)$$

which is allowed due to the gauge invariance  $\psi_{\mu\nu} \rightarrow \psi_{\mu\nu} + \xi_{[\mu,\nu]}$ . Ordinary Green’s function techniques may now be used to give the solution to (50) as

$$\mathbf{A}(x) = \frac{1}{4\pi} \int G(\mathbf{x} - \mathbf{y}) \mathbf{N}(\mathbf{y}) d^3y \quad (54)$$

where

$$G = \frac{1}{|\mathbf{x} - \mathbf{y}|} = \frac{1}{x} + \frac{\mathbf{x} \cdot \mathbf{y}}{x^3} + \dots \quad (55)$$

To lowest order (using only the  $1/x$  term in the expansion) gives

$$A_n = \frac{3k}{2\pi x} \int j^{0n} d^3y \quad (56)$$

where

$$j^{0n} = \frac{\eta^\mu}{2} \int d^2\zeta \delta^4(x - x(\zeta)) x_{,a}^0 x_{,b}^n \epsilon^{ab} \quad (57)$$

so

$$\int \frac{2}{\eta\mu} j^{0n} d^3y = \int_0^L dx^n = 0 \quad (58)$$

for closed strings. Now, using the next term in the expansion we have

$$A_n = \frac{\mathbf{x}}{4\pi x^3} \cdot \int \mathbf{y} N^n d^3y. \quad (59)$$

In these formulas  $\mathbf{y}$  is the body (string) centered coordinate, and represents the spatial part of  $\delta x^\mu$  used above. Using (47) the solution becomes

$$\mathbf{A} = \frac{3k}{8\pi} \frac{\mathbf{S} \times \mathbf{x}}{x^3}. \quad (60)$$

With this, the torsional field equations yield, letting  $\mathbf{r}$  replace  $\mathbf{x}$ , and considering the case that the spin points in the  $z$  direction,

$$\mathbf{b} = \frac{3k}{8\pi} \frac{S}{r^3} \left( 2 \cos(\theta) \hat{\mathbf{r}} + \sin(\theta) \hat{\boldsymbol{\theta}} \right). \quad (61)$$

The physical significance of this result is that the intrinsic spin, which results from the structure of the string, gives rise to a dipole field—the torsion field.

## 5 Conclusions

A string not only acts as the source of a gravitational field, it also becomes the source of torsion. In fact, when the string worldsheet area is coupled to the torsion potential, the physical property of the string that gives to torsion is intrinsic spin. The work presented here focused on closed strings only. The intrinsic spin of the string does not arise from motions of the string, but is due to the structure and spatial extent of the string. The Bianchi identity may be used to find the equation of motion of the string, and we found that the motion is geodesic only in lowest order.

## 6 Appendix

Above, the volume integral of the torsion source tensor was taken to vanish. This was done in (24). This can be shown to be true for two situations. One corresponds to the case that the string is a static rigid circle. It is noted that this assumption cannot hold for large cosmic strings under the influence of their own gravity. This result comes from the equation

of motion of string, which is obtained by taking variations with respect to  $x^\mu$ . However, this ‘geodesic’ postulate is not adopted here, the equation of motion of the string is obtained from the Bianchi identities, so this restriction does not hold. Allowing oscillations of the string nevertheless, we may then show that the time average of the integral will vanish. Assuming the period is very small, this is just as good as the rigid string assumption.

First, without any assumptions, one may see that, putting back the 3-volume element

$$\int \tilde{j}^{\sigma 0} d^3 x = 0. \quad (62)$$

From the definition  $j^{\mu\nu} \equiv (1/2)T^{[\mu\nu]}$  and (33) we have

$$j^{\sigma\nu} = \frac{\mu\eta}{2\sqrt{-g}} \int d^2\zeta \sqrt{-\gamma} \delta(x - x(\zeta)) x_{,a}^\sigma x_{,b}^\nu \epsilon^{ab}. \quad (63)$$

Integrating this over a volume one has,

$$\int \tilde{j}^{\sigma 0} d^3 x = \frac{\eta\mu}{2} \int_0^L d\zeta \frac{dx^\sigma}{d\zeta} = 0 \quad (64)$$

where again this holds for closed strings.

Now it is shown that  $\int \tilde{j}^{mn} d^3 x = 0$  for a rigid loop, or on average. For a rigid loop this becomes

$$\int \tilde{j}^{mn} d^3 x = -\frac{\eta\mu}{2} \int (y^m + \delta x^m) d \left( \frac{dx^n}{dx^0} \right) \quad (65)$$

after integration by parts. The first term vanishes since  $y^m$  goes to the center of mass and if  $d(\delta x^n)/dx^0 = 0$ , as it would for a rigid string, then the second term vanishes too, so that we have shown

$$\int \tilde{j}^{\mu\nu} = 0. \quad (66)$$

Alternatively, we may allow the string to undergo periodic oscillations in which case it is to be shown that

$$\langle \int j^{\mu\nu} d^3 x \rangle \equiv \frac{1}{T} \int dx^0 \int j^{\mu\nu} d^3 x = 0. \quad (67)$$

In this case we have

$$\langle \int j^{\mu\nu} d^3 x \rangle = \frac{\eta\mu}{T} \int dx^0 d\zeta \left( \frac{dx^\nu}{dx^0} \frac{dx^\mu}{d\zeta} - \frac{dx^\mu}{dx^0} \frac{dx^\nu}{d\zeta} \right) = 0. \quad (68)$$

Finally, one may see that the average over one period of the symmetric part of the energy momentum tensor does not vanish, and essentially gives back the energy momentum tensor, as we would expect.

Now we show the same kind of thing for the spatial part of the worldsheet metric. First, if we assume that the string is a rigid circular loop and, defining the  $\phi$  as an angular measure as  $\delta x^\mu$  traverses the loop, we get  $\gamma_{11} = -(\cos^2 \phi + \sin^2 \phi) = -1$ . On the other hand, if again we assume that there are periodic oscillations it is shown that  $\langle \gamma_{11} \rangle$  is approximately equal to -1. To see this, note that

$$\langle \gamma^{11} \rangle = \frac{1}{T} \int dx^0 \frac{dx^m}{d\zeta} \frac{dx^n}{d\zeta} g_{mn}. \quad (69)$$

Now use  $x^m = y^m + \delta x^m$  and assume further that  $\delta x^m$  may be written as  $\delta x^m = a^m + \epsilon^m(t)$  where  $a^m$  is time independent and  $\epsilon^m$  is periodic in  $T$ . Then,

$$\langle \gamma_{11} \rangle \approx \frac{1}{T} \int dx^0 \frac{d}{d\zeta} (a^m + \epsilon^m) \frac{d}{d\zeta} (a^n + \epsilon^n) \eta_{mn}. \quad (70)$$

Now, retaining terms to order  $\epsilon^m$  and using the periodicity of  $\epsilon^m$  we have

$$\langle \gamma_{11} \rangle = \frac{da^m}{d\zeta} \frac{da^n}{d\zeta} \eta_{mn} + \frac{2}{T} \frac{da^m}{d\zeta} \frac{d}{d\zeta} \int_0^T dx^0 \epsilon^n \eta_{mn}. \quad (71)$$

The second terms is zero due to the periodicity and the first terms replicates the result for the rigid loop, so we have

$$\langle \gamma^{11} \rangle = -1. \quad (72)$$

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